

# LOCAL AUTOMORPHISMS OF THE SETS OF STATES AND EFFECTS ON A HILBERT SPACE

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ABSTRACT. We prove that every local automorphism (affine 1-local, or non-affine 2-local) of the sets of all states on a Hilbert space is an automorphism. We also present similar results concerning the various automorphisms of the set of all effects.

## 1. INTRODUCTION

In what follows  $H$  denotes a separable infinite dimensional complex Hilbert space and  $B(H)$  stands for the algebra of all bounded linear operators on  $H$ .

The set  $\mathbb{S}$  of all states on  $H$  is defined as the set of all positive operators on  $H$  with trace 1, that is,

$$\mathbb{S} = \{T \in B(H) : T \geq 0, \operatorname{tr} T = 1\},$$

where  $\operatorname{tr}$  denotes the usual trace functional on  $B(H)$ .

The operator interval  $\mathbb{E} = [0, I]$  of all positive operators on  $H$  which are bounded by the identity  $I$  is called the effect algebra on  $H$ . The elements of  $\mathbb{E}$  are called effects.

These sets of operators, that is, the set of all states and the set of all effects, play important role in the mathematical description of quantum mechanics (see, for example, [3, 7] and the references therein as well). Just as with any algebraic structure, the study of the automorphisms of these sets when equipped with certain algebraic structures is of considerable importance.

So, what algebraic structures can be given to the mentioned two sets? As for  $\mathbb{S}$ , it is a convex subset of  $B(H)$  and one can consider its affine automorphisms (also called  $\mathbb{S}$ -automorphisms [3] or mixture-automorphisms [7]) which are the bijective maps  $\phi : \mathbb{S} \rightarrow \mathbb{S}$  satisfying

$$(1) \quad \phi(\lambda T + (1 - \lambda)S) = \lambda\phi(T) + (1 - \lambda)\phi(S)$$

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for every  $T, S \in \mathbb{S}$  and  $0 \leq \lambda \leq 1$ . It is known (see, for example, [3]) that every such  $\phi$  is of the form

$$(2) \quad \phi(T) = UTU^* \quad (T \in \mathbb{S}),$$

where  $U$  is either a unitary or an antiunitary operator on  $H$  (this latter notion means a conjugate-linear norm-preserving bijection of  $H$ ). As for the set of all effects, we have several possibilities how to introduce algebraic structures on  $\mathbb{E}$ . First,  $\mathbb{E}$  is a convex subset of  $B(H)$  and hence, just as in the case of  $\mathbb{S}$ , one can consider the affine automorphisms  $\phi$  of  $\mathbb{E}$ . In fact, in the literature it is very common that, in addition to that  $\phi$  is an affine bijection, they also assume that  $\phi$  is positive homogeneous as well (that is,  $\phi$  satisfies  $\phi(\lambda E) = \lambda\phi(E)$  for all  $E \in \mathbb{E}$  and  $0 \leq \lambda \leq 1$ ). In a recent paper [11] Molnár has determined all the affine automorphisms of the effect algebra in a von Neumann factor without this extra condition on homogeneity. As a particular case of his result we obtain that if  $\phi$  is an affine bijection of  $\mathbb{E}$ , then  $\phi$  is either of the form

$$(3) \quad \phi(E) = UEU^* \quad (E \in \mathbb{E}),$$

or of the form

$$(4) \quad \phi(E) = U(I - E)U^* \quad (E \in \mathbb{E}),$$

where in both cases  $U$  is either a unitary or an antiunitary operator on  $H$ . As a second possibility, there is also a so-called partial addition on  $\mathbb{E}$ . Namely, if  $E, F \in \mathbb{E}$  and  $E + F \in \mathbb{E}$ , then we define the sum of  $E$  and  $F$  by  $E + F$ . Now, we say that the bijective map  $\phi : \mathbb{E} \rightarrow \mathbb{E}$  is an effect-automorphism (in other terminology,  $\mathbb{E}$ -automorphism [3]) if

$$E + F \leq I \text{ if and only if } \phi(E) + \phi(F) \leq I$$

and in this case we have

$$\phi(E + F) = \phi(E) + \phi(F).$$

It is known [3], [7] that the effect-automorphisms are exactly the maps of the form

$$(5) \quad \phi(E) = UEU^* \quad (E \in \mathbb{E}),$$

where  $U$  is either a unitary or an antiunitary operator on  $H$ . Finally,  $\mathbb{E}$  has also a multiplicative structure, namely one can consider the product  $ABA$  on it ( $A, B \in \mathbb{E}$ ). This operation on  $\mathbb{E}$  was introduced in [11] and it was proved there that if  $\dim H \geq 3$ , then the automorphisms of  $\mathbb{E}$  with respect to this multiplication are the same as in (5).

The aim of this paper is to present some results on the local behaviour of the automorphisms of the sets of states and effects appearing above. There are relatively new investigations on local maps of operator algebras [5, 6, 10, 14] (also see the references therein). A linear map  $\psi$  on an algebra is called a linear (1-)local automorphism if at every point of the underlying algebra  $\psi$  coincides with an automorphism of the algebra (which automorphism may,

of course, differ from point to point). It is a remarkable fact concerning the algebra in question if it then follows that  $\psi$  is necessarily an automorphism. Following the concept of 2-local automorphisms due to Šemrl [16], we say that a map  $\psi$  on any algebraic structure (no linearity is assumed) is a 2-local automorphism if at every pair of points of the underlying structure  $\psi$  coincides with an automorphism (which automorphism may depend on the pair of points in question). Hence, we drop the assumption on the linearity of  $\psi$ , but instead we require that it behaves as an automorphism at every pair of points. Here, the problem is that whether a 2-local automorphism  $\psi$  is necessarily an automorphism? Recent result of that kind can be found in [13], [15]. If every linear (1-)local automorphism of an algebra is an automorphism or every 2-local automorphism of an algebraic structure is an automorphism, then we can say that the automorphisms of those structures are, in a certain sense, completely determined by their local actions.

The aim of this paper is to obtain results of that kind concerning the automorphism groups of  $\mathbb{S}$  and  $\mathbb{E}$ . We note that there are several other algebraic structures which also appear in relation to the mathematical formulations of quantum mechanics. These are, for instance, the orthomodular poset of all projections on  $H$ , the Jordan algebra of all bounded self-adjoint operators on  $H$  and the  $C^*$ -algebra of all bounded operators  $B(H)$ . The local automorphisms of these latter structures were investigated in [2], [13], [16].

## 2. RESULTS

Our first result which follows says that every affine (1-)local automorphism of  $\mathbb{S}$  is an automorphism.

**Theorem 1.** *Let  $\phi : \mathbb{S} \rightarrow \mathbb{S}$  be an affine transformation (that is, a function satisfying (1)) with the property that for every  $T \in \mathbb{S}$  there is an affine automorphism  $\phi_T$  of  $\mathbb{S}$  such that  $\phi(T) = \phi_T(T)$ . Then  $\phi$  is an affine automorphism of  $\mathbb{S}$ .*

*Proof.* Denote by  $C_1(H)$  the Banach algebra of all trace-class operators on  $H$ . Since  $\phi$  is an affine transformation on  $\mathbb{S}$ , it can be uniquely extended to a linear transformation  $\Phi : C_1(H) \rightarrow C_1(H)$ . In fact, if  $A$  is a nonzero positive trace-class operator, then let

$$\Phi_1(A) = \text{tr}(A)\phi\left(\frac{A}{\text{tr}(A)}\right).$$

We set  $\Phi_1(0) = 0$ . If  $B \in C_1(H)$  is selfadjoint, then  $B = B^+ - B^-$ , where

$$B^+ = \frac{B + |B|}{2} \in C_1(H), \quad B^- = \frac{|B| - B}{2} \in C_1(H)$$

are the positive and negative parts of  $B$ , respectively. Define

$$\Phi_2(B) = \Phi_1(B^+) - \Phi_1(B^-).$$

If  $C$  is an arbitrary trace-class operator, then  $C = C_1 + iC_2$ , where  $C_1, C_2$  are the real and the imaginary parts of  $C$  respectively, both of which belong to  $C_1(H)$ . We finally set

$$\Phi(C) = \Phi_2(C_1) + i\Phi_2(C_2).$$

It needs only elementary calculations to verify that  $\Phi$  is a linear extension of  $\phi$ . Since  $\phi$  is a local automorphism of  $\mathbb{S}$ , by (2) we obtain that  $\Phi$  is a positive linear map on  $C_1(H)$  sending rank-one projections to rank-one projections. The form of such maps can be easily derived from [4, Theorem 3.1, p. 21]. Namely, it follows from that result that either there is an isometry  $U$  on  $H$  such that

$$\Phi(A) = UAU^* \quad (A \in C_1(H)),$$

or there is an antiisometry (that is, a conjugate-linear norm-preserving map)  $V$  on  $H$  such that

$$\Phi(A) = VA^*V^* \quad (A \in C_1(H)),$$

or there is a rank-one projection  $P$  on  $H$  such that

$$\Phi(A) = (\text{tr } A)P \quad (A \in C_1(H)).$$

Using the local property of  $\phi$  we can easily infer that this third possibility is untenable. Without serious loss of generality we can assume that there exists an antiisometry  $V$  on  $H$  such that

$$(6) \quad \phi(T) = VTV^* \quad (T \in \mathbb{S}).$$

Pick an element  $T$  of  $\mathbb{S}$  with dense range. By the local form of  $\phi$  we see that  $\phi(T)$  must have dense range too. This gives us that the operator  $V$  in (6) is in fact antiunitary. This completes the proof of our assertion.  $\square$

The next statement asserts that every 2-local automorphism of the space of all states is an automorphism. We note that in the proof  $\langle \cdot, \cdot \rangle$  denotes the inner product on  $H$ . Because of our mathematical education we suppose that this form is linear in the first variable and conjugate-linear in the second variable. For any  $x, y \in H$ , the symbol  $|x\rangle\langle y|$  stands for the operator defined by  $|x\rangle\langle y|z = \langle z, y\rangle x$  ( $z \in H$ ).

**Theorem 2.** *Let  $\phi : \mathbb{S} \rightarrow \mathbb{S}$  be any transformation with the property that for every  $T, S \in \mathbb{S}$  there is an affine automorphism  $\phi_{T,S}$  of  $\mathbb{S}$  such that  $\phi(T) = \phi_{T,S}(T)$  and  $\phi(S) = \phi_{T,S}(S)$ . Then  $\phi$  is an affine automorphism of  $\mathbb{S}$ .*

*Proof.* Let  $\phi : \mathbb{S} \rightarrow \mathbb{S}$  be as above and let  $P, Q$  be rank-one projections on  $H$ . The local property of  $\phi$  implies that there exists an either unitary or antiunitary operator  $U$  on  $H$  such that

$$\phi(P) = UPU^*, \quad \phi(Q) = UQU^*.$$

In both cases we obtain that

$$(7) \quad \text{tr}(\phi(P)\phi(Q)) = \text{tr}(UPU^*UQU^*) = \text{tr}(UPQU^*) = \text{tr}(PQ).$$

By the form of the automorphisms of  $\mathbb{S}$  we see that the restriction of  $\phi$  onto the set  $P_1(H)$  of all rank-one projections on  $H$  maps  $P_1(H)$  into itself and by (7) it preserves the so-called transition probabilities (see [3, 2.1, p. 923]). We now apply a non-surjective variant of Wigner's classical theorem on transformations of  $P_1(H)$  preserving transition probabilities. According to the main result in [17] there is either an isometry or an antiisometry  $V$  on  $H$  such that

$$(8) \quad \phi(P) = VPV^* \quad (P \in P_1(H)).$$

We should remark here that the result in [17] applied above is about vector-to-vector transformations preserving the modulus of the inner product between vectors. But, just as with the original Wigner's theorem, it is easy to see that this is just an equivalent formulation of our problem concerning transformations on  $P_1(H)$  which preserve the transition probabilities.

We next show that  $V$  is surjective. Let  $(\lambda_n)$  be a sequence of pairwise different positive numbers with sum 1. Pick an orthonormal basis in  $H$  and let  $P_n$  denote the rank-one projection onto the subspace of  $H$  generated by the  $n^{\text{th}}$  basis vector. We fix the operator

$$T = \sum_n \lambda_n P_n.$$

By the local property of  $\phi$  we can clearly assume that  $\phi(T) = T$  for this particular  $T$ . Now, let  $n_0 \in \mathbb{N}$  be arbitrary. By the local property of  $\phi$  once again, we can choose a unitary or antiunitary operator  $U$  on  $H$  such that

$$\sum_n \lambda_n P_n = \phi\left(\sum_n \lambda_n P_n\right) = U \cdot \sum_n \lambda_n P_n \cdot U^* = \sum_n \lambda_n U P_n U^*$$

and

$$\phi(P_{n_0}) = U P_{n_0} U^*.$$

It follows from the first equation, that  $U P_n U^* = P_n$  for every  $n \in \mathbb{N}$ , so  $\phi(P_{n_0}) = P_{n_0}$ . Since  $n_0$  was arbitrary, we can infer that  $\phi(P_n) = P_n$  ( $n \in \mathbb{N}$ ). But this latter equality implies that the range of  $V$  in (8) contains the range of all  $P_n$ , that is, it contains an orthonormal basis and this gives us that  $V$  is surjective.

It remains to prove that  $\phi(S) = V S V^*$  holds for every  $S \in \mathbb{S}$ . Clearly, without serious loss of generality we can assume that  $V$  above is unitary. Let  $S \in \mathbb{S}$  and pick an arbitrary unit vector  $x \in H$ . Taking into account the form (8) of  $\phi$  on  $P_1(H)$  and the local property of  $\phi$ , we obtain on the one hand that

$$\begin{aligned} \phi(|x\rangle\langle x|)\phi(S)\phi(|x\rangle\langle x|) &= (V \cdot |x\rangle\langle x| \cdot V^*)\phi(S)(V \cdot |x\rangle\langle x| \cdot V^*) = \\ &= \langle \phi(S)Vx, Vx \rangle \cdot |Vx\rangle\langle Vx| \end{aligned}$$

and, on the other hand, that there is an either unitary or antiunitary operator  $W$  on  $H$  such that

$$\phi(|x\rangle\langle x|)\phi(S)\phi(|x\rangle\langle x|) = (W \cdot |x\rangle\langle x| \cdot W^*)(W S W^*)(W \cdot |x\rangle\langle x| \cdot W^*) =$$

$$\langle WSW^*Wx, Wx \rangle \cdot |Wx\rangle\langle Wx| = \langle Sx, x \rangle \cdot |Wx\rangle\langle Wx|.$$

This gives us that  $\langle V^*\phi(S)Vx, x \rangle = \langle Sx, x \rangle$ . Since this holds for every unit vector  $x \in H$ , we conclude that  $V^*\phi(S)V = S$ . Therefore, we have

$$\phi(S) = VSV^*$$

for every  $S \in \mathbb{S}$  and this completes the proof.  $\square$

Since on  $\mathbb{E}$  there are several algebraic structures we decided to begin with the 2-local automorphisms.

**Theorem 3.** *Let  $\phi : \mathbb{E} \rightarrow \mathbb{E}$  be a transformation with the property that for every  $E, F \in \mathbb{E}$  there exists an either unitary or antiunitary operator  $U_{E,F}$  on  $H$  such that  $\phi(E) = U_{E,F}EU_{E,F}^*$  and  $\phi(F) = U_{E,F}FU_{E,F}^*$ . Then there exists an either unitary or antiunitary operator  $U$  on  $H$  such that*

$$\phi(E) = UEU^* \quad (E \in \mathbb{E}).$$

*Proof.* It is clear that the restriction of  $\phi$  onto the orthomodular poset  $\mathbb{P}$  of all projections is a 2-local automorphism of  $\mathbb{P}$ . Now, [13, Proposition] tells us that every 2-local automorphism of  $\mathbb{P}$  is an automorphism. Therefore, there exists an either unitary or antiunitary operator  $U$  on  $H$  such that  $\phi(P) = UPU^*$  for every projection  $P$ . Clearly, we can suppose that  $\phi(P) = P$  for every  $P \in \mathbb{P}$ . By the 2-local property of  $\phi$  we easily deduce  $\phi(\lambda P) = \lambda P$  ( $\lambda \in [0, 1]$ ). This property also gives us that  $E \leq F$  if and only if  $\phi(E) \leq \phi(F)$ . Therefore, for any  $E \in \mathbb{E}$  we have  $\lambda P \leq E$  if and only if  $\lambda P = \phi(\lambda P) \leq \phi(E)$  ( $P \in \mathbb{P}$ ,  $\lambda \in [0, 1]$ ).

Now, let  $P_1, \dots, P_n$  be pairwise orthogonal spectral projections of  $E$  and  $\lambda_1, \dots, \lambda_n \in [0, 1]$  such that  $\sum_i \lambda_i P_i \leq E$ . Then we have  $\lambda_i P_i \leq \phi(E)$  for every  $i = 1, \dots, n$ . As  $P_i$  commutes with  $E$ , by the 2-local property of  $\phi$  we obtain that  $P_i = \phi(P_i)$  commutes with  $\phi(E)$ . Multiplying the inequality  $\lambda_i P_i \leq \phi(E)$  by  $P_i$  from both sides, we have  $\lambda_i P_i \leq P_i \phi(E) P_i$ . By the commutativity of  $P_i$  and  $\phi(E)$  we have that  $P_i$  and  $\sqrt{\phi(E)}$  also commute. We can compute

$$\begin{aligned} \sum_i \lambda_i P_i &\leq \sum_i P_i \phi(E) P_i = \sum_i P_i \phi(E) = \\ &\sqrt{\phi(E)} \left( \sum_i P_i \right) \sqrt{\phi(E)} \leq \sqrt{\phi(E)} I \sqrt{\phi(E)} = \phi(E). \end{aligned}$$

Approximating  $E$  with operators of the form  $\sum_i \lambda_i P_i$  we obtain that  $E \leq \phi(E)$ . Interchanging the role of  $E$  and  $\phi(E)$  in the argument above, we also get  $\phi(E) \leq E$ . Therefore,  $\phi(E) = E$  for every  $E \in \mathbb{E}$  and this completes the proof.  $\square$

We now treat the various (1-)local automorphisms of  $\mathbb{E}$ .

**Theorem 4.** *Let  $\phi : \mathbb{E} \rightarrow \mathbb{E}$  be an affine transformation with the property that for every  $E \in \mathbb{E}$  there exists an affine automorphism  $\phi_E$  of  $\mathbb{E}$  such that  $\phi(E) = \phi_E(E)$ . Then  $\phi$  is an affine automorphism of  $\mathbb{E}$ .*

*Proof.* By the form of the affine automorphisms of  $\mathbb{E}$  it follows that either  $\phi(0) = 0$  or  $\phi(0) = I$ . Suppose first that  $\phi(0) = 0$ . Then one can prove just as in the proof of Theorem 1 that  $\phi$  can be extended to a linear transformation  $\Phi : B(H) \rightarrow B(H)$ . By the local form of  $\phi$  we deduce that  $\Phi$  sends projections to projections. Since  $\Phi$  is a positive linear transformation, it is well-known to be norm-continuous. Now, it needs only elementary computation to verify that  $\Phi$  is a Jordan  $*$ -endomorphism of  $B(H)$  (see, for example, the proof of [8, Theorem 2]) We show that  $\phi$  preserves the rank-one projections. Let  $P$  be a rank-one projection. Suppose that  $\phi(P) = U(I - P)U^*$  for some unitary or antiunitary operator  $U$  on  $H$ . As for  $\phi((1/2)P)$  we have two possibilities. First assume that there exists an either unitary or antiunitary operator  $W$  such that  $\phi((1/2)P) = W((1/2)P)W^*$ . Since  $\phi((1/2)P) = (1/2)\phi(P)$  (this follows from the fact that  $\phi$  is affine and  $\phi(0) = 0$ ), we have

$$(1/2)U(I - P)U^* = W((1/2)P)W^*$$

which is an obvious contradiction if one considers the ranks of the operators appearing in that equality. So, it remains that  $\phi((1/2)P) = W(I - (1/2)P)W^*$  for some unitary or antiunitary operator  $W$ . This leads to the equality

$$(1/2)U(I - P)U^* = W(I - (1/2)P)W^*$$

which also leads to a contradiction since the spectrum of the operator on the left-hand side is  $\{0, 1/2\}$  while the spectrum of the operator on the right-hand side is  $\{1/2, 1\}$ . Therefore,  $\phi(P)$  must be of the form  $UPU^*$  for some unitary or antiunitary operator  $U$  and this gives us that  $\Phi$  preserves the rank-one projection. Moreover, similarly as above one can verify that  $\phi(I) = I$  and this yields that  $\Phi(I) = I$ . So  $\Phi$  is a Jordan  $*$ -endomorphism of  $B(H)$  which preserves the rank-one projections and it sends  $I$  to  $I$ . Similarly as in the proof of [1, Theorem 3] we obtain that  $\Phi$  is implemented by either a unitary or an antiunitary operator. This completes the proof in the case when  $\phi(0) = 0$ . If  $\phi(0) = I$ , then one can consider the transformation  $E \mapsto I - \phi(E)$  to reduce the proof to the previous case.  $\square$

**Theorem 5.** *Let  $\phi : \mathbb{E} \rightarrow \mathbb{E}$  be an additive transformation in the sense that for every  $E, F \in \mathbb{E}$  with  $E + F \in \mathbb{E}$  we have  $\phi(E + F) = \phi(E) + \phi(F)$ . If for every  $E \in \mathbb{E}$  there exists an effect-automorphism  $\phi_E$  of  $\mathbb{E}$  such that  $\phi(E) = \phi_E(E)$ , then  $\phi$  is an effect-automorphism.*

*Proof.* We are going to prove that  $\phi$  is an affine transformation and  $\phi(0) = 0$ . If this is done, then one can simply refer to our previous theorem.

The fact that  $\phi(0) = 0$  is trivial by the local property of  $\phi$ . From the additivity of  $\phi$  it follows that  $\phi$  is monotone increasing. Once again, the additivity of  $\phi$  gives us that  $n\phi(E) = \phi(nE)$  for every  $n \in \mathbb{N}$  and  $E \in \mathbb{E}$  with  $nE \in \mathbb{E}$ . Now simple calculation shows that  $r\phi(E) = \phi(rE)$  for every positive rational number  $r$  and  $E \in \mathbb{E}$  with  $rE \in \mathbb{E}$ . Finally, for any real

number  $0 < \alpha < 1$  and  $E \in \mathbb{E}$  we have  $\phi(\alpha E) = \alpha\phi(E)$ . Indeed, we can construct two sequences  $(\beta_n), (\gamma_n)$  of rational numbers in  $[0, 1]$  such that,  $\beta_n < \alpha < \gamma_n$  for every  $n$  and both  $(\beta_n)$  and  $(\gamma_n)$  converge to  $\alpha$ . Using the monotonicity of  $\phi$ , we get

$$\beta_n\phi(E) = \phi(\beta_n E) \leq \phi(\alpha E) \leq \phi(\gamma_n E) = \gamma_n\phi(E),$$

and taking limits we have the desired equality  $\alpha\phi(E) = \phi(\alpha E)$ . If  $E, F \in \mathbb{E}$  are arbitrary and  $\lambda \in [0, 1]$ , then we can compute

$$\phi(\lambda E + (1 - \lambda)F) = \phi(\lambda E) + \phi((1 - \lambda)F) = \lambda\phi(E) + (1 - \lambda)\phi(F).$$

This shows that  $\phi$  is affine and we are done.  $\square$

**Theorem 6.** *Let  $\phi : \mathbb{E} \rightarrow \mathbb{E}$  be a transformation satisfying*

$$\phi(EFE) = \phi(E)\phi(F)\phi(E) \quad (E, F \in \mathbb{E}).$$

*If for every  $E \in \mathbb{E}$  there exists an automorphism  $\phi_E$  of  $\mathbb{E}$  of the form (5) such that  $\phi(E) = \phi_E(E)$ , then  $\phi$  is an automorphism of  $\mathbb{E}$  of the form (5).*

*Proof.* First we show that  $\phi$  preserves the projections and the orthogonality between them. Indeed, by the local property of  $\phi$  we deduce that  $\phi$  sends projections to projections. Let  $P, Q \in B(H)$  be projections such that  $PQ = 0$ . Then we have  $0 = \phi(PQP) = \phi(P)\phi(Q)\phi(P)$  which implies that

$$0 = \phi(P)\phi(Q)\phi(P) = (\phi(Q)\phi(P))^*(\phi(Q)\phi(P)).$$

This gives us that  $\phi(Q)\phi(P) = 0$ .

Next observe that  $\phi$  also preserves the partial ordering  $\leq$  between projections. To see this, let  $P, Q \in B(H)$  be projections and suppose that  $P \leq Q$ . Then we have  $P = QPQ$  and  $\phi(P) = \phi(QPQ) = \phi(Q)\phi(P)\phi(Q)$  which yields that the range of  $\phi(P)$  is included in the range of  $\phi(Q)$ , that is,  $\phi(P) \leq \phi(Q)$ .

We prove that  $\phi$  is finitely orthoadditive on the set of all finite rank projections in  $B(H)$ . To see this, let  $P, Q \in P(H)$  be mutually orthogonal finite rank projections. By the order preserving property of  $\phi$  we have  $\phi(P), \phi(Q) \leq \phi(P + Q)$ . Since  $\phi$  preserves orthogonality, we infer that  $\phi(P) + \phi(Q)$  is a projection and  $\phi(P) + \phi(Q) \leq \phi(P + Q)$ . By the local form of  $\phi$  we find that  $\phi$  also preserves the rank of projections. Hence we compute

$$\text{rank}(\phi(P) + \phi(Q)) = \text{rank}(\phi(P)) + \text{rank}(\phi(Q)) =$$

$$\text{rank}(P) + \text{rank}(Q) = \text{rank}(P + Q) = \text{rank}(\phi(P + Q))$$

which yields that  $\phi(P) + \phi(Q) = \phi(P + Q)$ .

We now extend  $\phi$  from the set of all finite rank projections to a Jordan  $*$ -homomorphism of the algebra  $F(H)$  of all finite rank operators in  $B(H)$ . This can be done in a way very similar to what was followed in the proof of [9, Theorem 1]. By the properties of  $\phi$ , this extension preserves the rank-one



projections and the orthogonality between them. Now, [12, Lemma 2] tells us that there exists an isometry or antiisometry  $U : H \rightarrow H$  such that

$$(9) \quad \phi(P) = UPU^*$$

holds for every finite rank projection  $P$  on  $H$ .

If  $B \in [0, I]$  is an arbitrary operator, due to  $\phi(I) = I$  we have  $(\phi(B))^2 = \phi(B)\phi(I)\phi(B) = \phi(B^2)$ , which implies that  $\sqrt{\phi(A)} = \phi(\sqrt{A})$  holds for every  $A \in [0, I]$ .

We prove that  $\phi$  is homogeneous in some sense. Let  $P \in B(H)$  be a finite rank projection and  $\lambda \in [0, 1]$ . By the local property of  $\phi$  we get  $\phi(P) = VPV^*$  and  $\phi(\lambda P) = W\lambda PW^* = \lambda WPW^*$ , where  $V$  and  $W$  are unitary or antiunitary operators. Since  $\lambda P \leq P$ , it follows that  $\phi(\lambda P) \leq \phi(P)$ . Therefore,  $\lambda WPW^* \leq VPV^*$ . As  $WPW^*$  and  $VPV^*$  are projections of the same rank, it follows that  $WPW^* = VPV^*$  which implies that  $\phi(\lambda P) = \lambda\phi(P)$ .

We claim that the operator  $U$  appearing in (9) is either unitary or antiunitary. To verify this, let  $P_i$  ( $i \in \mathbb{N}$ ) be pairwise orthogonal rank-one projections and  $\lambda_i \in [0, 1]$  ( $i \in \mathbb{N}$ ) be scalars, such that  $\sum_{i=1}^{\infty} P_i = I$  and  $\sum_{i=1}^{\infty} \lambda_i < \infty$ . Set  $A = \sum_{i=1}^{\infty} \lambda_i P_i$ . By the properties of  $\phi$  what we already know, we infer

$$\begin{aligned} \sum_{i=1}^n \lambda_i \phi(P_i) &= \sum_{i=1}^n \phi(\lambda_i P_i) = \sum_{i=1}^n \phi(\sqrt{A} P_i \sqrt{A}) = \sum_{i=1}^n \phi(\sqrt{A}) \phi(P_i) \phi(\sqrt{A}) \\ &= \sqrt{\phi(A)} \sum_{i=1}^n \phi(P_i) \sqrt{\phi(A)} \leq \sqrt{\phi(A)} I \sqrt{\phi(A)} = \phi(A). \end{aligned}$$

So we have  $\sum_{i=1}^{\infty} \lambda_i \phi(P_i) \leq \phi(A)$ . By the local property of  $\phi$  we get that

$$\text{tr}(\phi(A)) = \text{tr}(A) = \text{tr}\left(\sum_{i=1}^{\infty} \lambda_i P_i\right) = \sum_{i=1}^{\infty} \lambda_i = \text{tr}\left(\sum_{i=1}^{\infty} \lambda_i \phi(P_i)\right),$$

which implies that  $\text{tr}(\phi(A) - \sum_{i=1}^{\infty} \lambda_i \phi(P_i)) = 0$ . Since if the trace of a positive operator is 0 then the operator in question is necessarily 0, it follows that

$$\sum_{i=1}^{\infty} \lambda_i \phi(P_i) = \phi(A).$$

Therefore, we have

$$\phi(A) = \sum_{i=1}^{\infty} \lambda_i U P_i U^* = U \left( \sum_{i=1}^{\infty} \lambda_i P_i \right) U^* = U A U^*.$$

Since, by the local property of  $\phi$ , the range of  $\phi(A)$  must be dense, we deduce that  $U$  is a unitary or antiunitary operator.

Clearly, without any loss of generality we now can assume that  $\phi(P) = P$  holds for every finite rank projection  $P$ . We claim that  $\phi$  is the identity on

$\mathbb{E}$ . Let  $A \in \mathbb{E}$ . Pick an arbitrary rank-one projection  $P$ . Then  $PAP = \lambda P$  holds for some  $\lambda \in [0, 1]$ . We compute

$$PAP = \lambda P = \lambda\phi(P) = \phi(\lambda P) = \phi(PAP) = \phi(P)\phi(A)\phi(P) = P\phi(A)P.$$

Since  $P$  was arbitrary, we obtain  $\phi(A) = A$  for every  $A \in \mathbb{E}$ . This completes the proof.  $\square$

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